

The effects of curvature correction terms on brane cosmology

E. Papantonopoulos*

Department of Physics, National Technical University of Athens,
Zografou Campus GR 157 73, Athens, Greece

Abstract

We study the cosmology of the Randall-Sundrum brane-world where the Einstein-Hilbert action is modified by curvature correction terms: a four-dimensional scalar curvature from induced gravity on the brane, and a five-dimensional Gauss-Bonnet curvature term. The combined effect of these curvature corrections to the action removes the infinite-density big bang singularity, although the curvature can still diverge for some parameter values. A radiation brane undergoes accelerated expansion near the minimal scale factor, for a range of parameters. This acceleration is driven by the geometric effects, without an inflaton field or negative pressures. At late times, conventional cosmology is recovered.

* Invited talk given at Tenth Marcel Grossmann Meeting, Rio de Janeiro, July 20-26, 2003; lpapa@central.ntua.gr

1 Introduction

The Randall-Sundrum II model [1] provides a simple phenomenology for exploring brane-world gravity and associated ideas from string theory. Matter and gauge interactions are localized on the brane, while gravity accesses the infinite extra dimension, but is localized at low energies due to the warping (curvature) of the extra dimension. The cosmological generalization of the Randall-Sundrum model is characterized by an unconventional evolution at early times, while standard cosmology is recovered at late times [2, 3].

The Randall-Sundrum model is based on the Einstein-Hilbert action in five dimensions. This gravitational action can be generalized in various ways. Two important generalizations have been considered recently. The first is a four-dimensional scalar curvature term in the brane action. This induced gravity correction arises because the localized matter fields on the brane, which couple to bulk gravitons, can generate via quantum loops a localized four-dimensional world-volume kinetic term for gravitons [4, 5]. The second is a Gauss-Bonnet correction to the five-dimensional action. This gives the most general action with second-order field equations in five dimensions [6]. Furthermore, in an effective action approach to string theory, the Gauss-Bonnet term corresponds to the leading order quantum corrections to gravity, and its presence guarantees a ghost-free action [7].

Here, we investigate the effects of the combined curvature corrections, from both induced gravity and Gauss-Bonnet. In some sense, these are the leading-order corrections to the gravitational action, and there is no obvious way to argue that one effect is dominant over the other. Indeed, the corrections operate at different energy levels. Induced gravity introduces intriguing late-time modifications, which can accelerate the universe even in the absence of dark energy [8, 9]. If there is a brane tension on the brane and a cosmological constant in the bulk, there are further modifications [4, 11, 12, 9] with some interesting astrophysical implications [13]. However, one expects that string-theory type modifications to the Einstein-Hilbert action must also operate at early times, and so it is sensible to incorporate the Gauss-Bonnet correction.

At early times, the Randall-Sundrum model gives an unconventional cosmology, with the Hubble rate H scaling as ρ , rather than $\rho^{1/2}$ as in general relativity. The Gauss-Bonnet correction to this picture changes the ρ dependence of H to $\rho^{2/3}$, and therefore an infinite-density big bang is encountered, as in the Randall-Sundrum case. The combined effect of Gauss-Bonnet and induced gravity modifications [14] eliminates the infinite-density solutions, because the scale factor is bounded. However, the initial curvature

may diverge since there is a range of parameters for which the solutions start their evolution with infinite acceleration. In the low-energy regime of these solutions, the standard cosmology is recovered (with positive Newton constant).

2 Friedmann equation on the brane

The total gravitational action is

$$\begin{aligned}
S_{\text{grav}} = & \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-(5)g} \left\{ (5)R - 2\Lambda_5 + \alpha \left[(5)R^2 \right. \right. \\
& \left. \left. - 4 (5)R_{AB} (5)R^{AB} + (5)R_{ABCD} (5)R^{ABCD} \right] \right\} \\
& + \frac{r}{2\kappa_5^2} \int_{y=0} d^4x \sqrt{-(4)g} \left[(4)R - 2\Lambda_4 \right], \tag{1}
\end{aligned}$$

where the Gauss-Bonnet coupling α has dimensions $(length)^2$ and is defined as

$$\alpha = \frac{1}{8g_s^2}, \tag{2}$$

with g_s the string energy scale, while the induced-gravity crossover length scale is

$$r = \frac{\kappa_5^2}{\kappa_4^2} = \frac{M_4^2}{M_5^3}. \tag{3}$$

Here, the fundamental (M_5) and the four-dimensional (M_4) Planck masses are given by

$$\kappa_5^2 = 8\pi G_5 = M_5^{-3}, \quad \kappa_4^2 = 8\pi G_4 = M_4^{-2}. \tag{4}$$

The brane tension is given by

$$\lambda = \frac{\Lambda_4}{\kappa_4^2}, \tag{5}$$

and is non-negative.

We assume there are no sources in the bulk other than Λ_5 . Varying Eq. (1) with respect to the bulk metric $(5)g_{AB}$, we obtain the field equations:

$$(5)G_{AB} - \frac{\alpha}{2} \left[(5)R^2 - 4 (5)R_{CD} (5)R^{CD} \right]$$

$$\begin{aligned}
& + {}^{(5)}R_{CDEF} {}^{(5)}R^{CDEF} \Big] {}^{(5)}g_{AB} \\
& + 2\alpha \left[{}^{(5)}R {}^{(5)}R_{AB} - 2 {}^{(5)}R_{AC} {}^{(5)}R_B{}^C \right. \\
& \quad \left. - 2 {}^{(5)}R_{ACBD} {}^{(5)}R^{CD} + {}^{(5)}R_{ACDE} {}^{(5)}R_B{}^{CDE} \right] \\
& = -\Lambda_5 {}^{(5)}g_{AB} + \kappa_5^2 {}^{(\text{loc})}T_{AB} \hat{\delta}(y), \tag{6}
\end{aligned}$$

where ${}^{(4)}g_{AB} = {}^{(5)}g_{AB} - n_A n_B$ is the induced metric on the hypersurfaces $\{y = \text{constant}\}$, with n^A the normal vector. The localized energy-momentum tensor of the brane is

$${}^{(\text{loc})}T_{AB} \equiv {}^{(4)}T_{AB} - \lambda {}^{(4)}g_{AB} - \frac{r}{\kappa_5^2} {}^{(4)}G_{AB}, \tag{7}$$

and we have used the normalized Dirac delta function, $\hat{\delta}(y) = \sqrt{{}^{(4)}g/{}^{(5)}g} \delta(y)$. The pure Gauss-Bonnet correction is the case $r = 0$, the pure induced gravity correction is the case $\alpha = 0$, and the Randall-Sundrum case is $r = 0 = \alpha$.

For a homogeneous and isotropic brane at fixed coordinate position $y = 0$ in the bulk, we get a generic cubic equation in H^2

$$\begin{aligned}
& \frac{4}{r^2} \left[1 + \frac{8}{3} \alpha \left(H^2 + \frac{k}{a^2} + \frac{\Phi_0}{2} \right) \right]^2 \left(H^2 + \frac{k}{a^2} - \Phi_0 \right) \\
& = \left[H^2 + \frac{k}{a^2} - \frac{\kappa_4^2}{3} (\rho + \lambda) \right]^2, \tag{8}
\end{aligned}$$

where $\Phi_0 = \Phi(t, 0)$ and Φ is a solution of the equation $\Phi + 2\alpha\Phi^2 = \Lambda_5/6 + \mathcal{C}/a^4$ with \mathcal{C} is an integration constant, from which the Friedmann equations of all known braneworld models can be derived.

In the limit $r \rightarrow 0$, Eq. (8) becomes

$$\begin{aligned}
& \left[1 + \frac{8}{3} \alpha \left(H^2 + \frac{k}{a^2} + \frac{\Phi_0}{2} \right) \right]^2 \left(H^2 + \frac{k}{a^2} - \Phi_0 \right) \\
& = \frac{\kappa_5^4}{36} (\rho + \lambda)^2. \tag{9}
\end{aligned}$$

The single real solution of this cubic which is compatible with the $\alpha \rightarrow 0$ limit of Eq. (9), is the Friedmann equation with Gauss-Bonnet correction [10]

$$H^2 + \frac{k}{a^2} = \frac{1}{8\alpha} \left(-2 + \frac{64I^2}{J} + J \right), \tag{10}$$

where the dimensionless quantities I, J are given by

$$I = \frac{1}{8}(1 + 4\alpha\Phi_0) = \pm \frac{1}{8} \left[1 + \frac{4}{3}\alpha\Lambda_5 + \frac{8\alpha\mathcal{C}}{a^4} \right]^{1/2}, \quad (11)$$

$$J = \left[\frac{\kappa_5^2 \sqrt{\alpha}}{\sqrt{2}}(\rho + \lambda) + \sqrt{\frac{\kappa_5^4 \alpha}{2}(\rho + \lambda)^2 + (8I)^3} \right]^{2/3}. \quad (12)$$

In the other limit, $\alpha \rightarrow 0$, Eq. (8) yields

$$\frac{4}{r^2} \left(H^2 + \frac{k}{a^2} - \Phi_0 \right) = \left[H^2 + \frac{k}{a^2} - \frac{\kappa_4^2}{3}(\rho + \lambda) \right]^2. \quad (13)$$

The solution is the Friedmann equation of the induced gravity model [4, 9, 11, 12]

$$\begin{aligned} H^2 + \frac{k}{a^2} &= \frac{\kappa_4^2}{3}(\rho + \lambda) + \frac{2}{r^2} \\ &\pm \frac{1}{\sqrt{3}r} \left[4\kappa_4^2(\rho + \lambda) - 2\Lambda_5 + \frac{12}{r^2} - \frac{12\mathcal{C}}{a^4} \right]^{1/2}. \end{aligned} \quad (14)$$

Finally taking both limits we find the Friedmann equation of the Randall-Sundrum model.

Returning to the general case of both curvature corrections, we need the real solution of Eq. (8) in the simplest possible form. We define the dimensionless parameter

$$\beta = \frac{256\alpha}{9r^2}, \quad (15)$$

and the dimensionless variables

$$P = 1 + 3\beta I, \quad (16)$$

$$Q = \beta \left[\frac{1}{4} + I + \frac{\kappa_4^2 \alpha}{3}(\rho + \lambda) \right], \quad (17)$$

$$X = \beta \left[\frac{1}{4} + I + \alpha \left(H^2 + \frac{k}{a^2} \right) \right]. \quad (18)$$

Then, Eq. (8) takes the form

$$X^3 - PX^2 + 2QX - Q^2 = 0. \quad (19)$$

The single real solution of this equation which is compatible with the $\alpha \rightarrow 0$ limit of Eq. (8), i.e. with Eq. (14), is

$$X = \frac{P}{3} - \frac{2}{3}\sqrt{P^2 - 6Q} \cos\left(\Theta \pm \frac{\pi}{3}\right), \quad (20)$$

where

$$\Theta(P, Q) = \frac{1}{3} \arccos \left[\frac{2P^3 + 27Q^2 - 18PQ}{2(P^2 - 6Q)^{3/2}} \right]. \quad (21)$$

This solution corresponds to the positive sign in Eq. (11), while the negative sign does not provide the correct $\alpha \rightarrow 0$ limit. The \pm sign in Eq. (20) is the same as that in Eq. (14). The region in (P, Q) -space for which Eq. (20) is defined, is

$$1 \leq P < \frac{4}{3}, \quad (22)$$

$$\begin{aligned} 2[9P - 8 - (4 - 3P)^{3/2}] &\leq 27Q \\ &\leq 3P[3 - \sqrt{3(3 - 2P)}]. \end{aligned} \quad (23)$$

Finally, we can write the Friedmann equation of the combined Gauss-Bonnet and induced gravity brane-world as [14]

$$\begin{aligned} H^2 + \frac{k}{a^2} &= \frac{4 - 3\beta}{12\beta\alpha} \\ &- \frac{2}{3\beta\alpha} \sqrt{P^2 - 6Q} \cos\left(\Theta \pm \frac{\pi}{3}\right). \end{aligned} \quad (24)$$

This has a very different structure than its limiting forms, Eqs. (10) and (14). A closed system of equations for the brane-world follows if we also consider the continuity equation,

$$\dot{\rho} + 3H\rho(1 + w) = 0, \quad (25)$$

where $w = p/\rho \geq -1$ and $\rho \geq 0$. If w is constant, then $\rho = \rho_0(a_0/a)^{3(1+w)}$, and we can choose $a_0 = 1$.

3 Cosmological dynamics

The dimensionless variable P is a function of I and carries the information of the bulk onto the brane, since by Eq. (11) it depends on the bulk cosmological constant Λ_5 and the mass \mathcal{C} of the bulk black hole. The dimensionless

variable Q includes information about the matter and energy content of the brane. These are the key variables determining the cosmological dynamics.

The four-dimensional scalar curvature term of the induced gravity and the Gauss-Bonnet term in the five-dimensional space are all curvature corrections to the Randall-Sundrum model. One could be led to expect that r^2 and α are of the same order. However, this is not necessarily true. The crossover scale r of the induced gravity appears in loops involving matter particles, and depending on the mass, it can be arbitrarily large. On the other hand, the Gauss-Bonnet coupling α arises from integrating out massive string modes, and depending on the scale of the theory, it can also be arbitrarily large.

3.1 No infinite-density big bang

An important feature arises from inequalities (22) and (23), which show that P and hence Q are bounded from above. Furthermore, Eqs. (16) and (22) show that I is bounded from above (and positive). Therefore, it follows from Eq. (17) that the energy density ρ cannot become infinite, which means that *an infinite-density singularity $a = 0$ is never encountered*:

$$a(t) \geq a_0 > 0, \quad \rho(t) \leq \rho_0 < \infty. \quad (26)$$

This is true independent of the spatial curvature k , or the equation of state.

This result is remarkable since the Gauss-Bonnet correction, which is expected to dominate at early times, on its own does not remove the infinite-density singularity [15, 10, 16], while the induced gravity correction on its own mostly affects the late-time evolution. However, the combination of these curvature corrections is effectively “nonlinear”, producing a result that is not obviously the superposition of their separate effects. In general terms, the early-universe behaviour is strongly modified by the effective coupling of the 5D curvature to the matter [15].

In the pure Gauss-Bonnet theory ($\mathcal{C} \geq 0$), the early-universe evolves from infinite density at $a = 0$. The Friedmann equation (10) for $\mathcal{C} = 0$, or for $\mathcal{C} > 0$, $w > 0$, is approximated by

$$H^2 + \frac{k}{a^2} \approx \left(\frac{\kappa_5^2}{16\alpha} \right)^{2/3} \rho^{2/3}. \quad (27)$$

For $w > 0$, or for $\mathcal{C} = 0 = k$, the density term dominates the curvature term, and

$$a \approx \text{const} \times t^{1/(1+w)}. \quad (28)$$

The Gauss-Bonnet correction causes the universe to expand faster relative to Einstein gravity, for which $a \propto t^{2/3(1+w)}$, and to the Randall-Sundrum model, for which $a \propto t^{1/3(1+w)}$. At the same time, a given energy density produces a smaller expansion rate in the Gauss-Bonnet case. This means that there is *less* Hubble friction for a given potential than in general relativity, so that slow-roll is more difficult to achieve. For the same reason, scalar perturbations generated during slow-roll inflation will have a smaller amplitude than those generated at the same energy density in general relativity. This is opposite to the Randall-Sundrum model [17].

3.2 Geometric inflation in a radiation universe

We assume $\mathcal{C} > 0$, i.e. there is a black hole present in the bulk. Defining the acceleration variable $f = \ddot{a}/a = \dot{H} + H^2$, we obtain from Eqs. (24) and (25) that

$$\begin{aligned} f = & \frac{4 - 3\beta}{12\beta\alpha} + \frac{\cos(\Theta \pm \pi/3)}{3\beta\alpha\sqrt{P^2 - 6Q}} \times \\ & \times \left[c_1 + \sigma(1 - 3w) |(P - 1)^2 - c_2|^{3(1+w)/4} \right. \\ & \left. + (P^2 - 6Q) \frac{\dot{\Theta}}{H} \tan\left(\Theta \pm \frac{\pi}{3}\right) \right], \end{aligned} \quad (29)$$

where

$$\begin{aligned} (P^2 - 6Q) \frac{\dot{\Theta}}{H} = & \frac{1}{\sqrt{3}\sqrt{4Q(9P - 8) - 4P^2(P - 1) - 27Q^2}} \times \\ & \times \left\{ 2(2P - 9Q) \left[(P - 1)^2 - c_2 \right] \right. \\ & \left. - 3\sigma(1 + w) [3Q - 2P(P - 1)] \times \right. \\ & \left. \times |(P - 1)^2 - c_2|^{3(1+w)/4} \right\}, \end{aligned} \quad (30)$$

and

$$c_1 = -2 + \beta(3 + 4\alpha\Lambda_4) - 3\beta^2(3 + 4\alpha\Lambda_5)/32, \quad (31)$$

$$c_2 = \frac{3}{32}\beta^2(3 + 4\alpha\Lambda_5), \quad (32)$$

$$\sigma = \beta\alpha\kappa_4^2\rho_0 \left(\frac{8}{9\beta^2\alpha\mathcal{C}} \right)^{3(1+w)/4}. \quad (33)$$

These equations are formidably complicated, and we do not attempt an exhaustive analysis. Instead, we show that for a radiation brane in the presence of a bulk black hole, there is a range of parameters for which there is inflationary expansion, $f > 0$, near a_0 .

For a radiation era, $w = \frac{1}{3}$,

$$f = \frac{4-3\beta}{12\beta\alpha} + \frac{\cos(\Theta \pm \pi/3)}{3\beta\alpha\sqrt{P^2-6Q}} \times \\ \times \left[c_1 + (P^2-6Q) \frac{\dot{\Theta}}{H} \tan\left(\Theta \pm \frac{\pi}{3}\right) \right], \quad (34)$$

where

$$(P^2-6Q) \frac{\dot{\Theta}}{H} = 2|(P-1)^2 - c_2| \times \\ \times \frac{4\sigma P^2 - 2(2\sigma-1)P - 3(2\sigma+3)Q}{\sqrt{3}\sqrt{4Q(9P-8) - 4P^2(P-1) - 27Q^2}}. \quad (35)$$

We assume that

$$\Lambda_5 > -\frac{3}{4\alpha}, \quad (36)$$

and define the additional parameters

$$P_1 = 1 + \sqrt{\frac{c_2}{2}}, \quad (37)$$

$$Q_1 = \frac{1}{12} \left(c_1 + c_2 + 2 + 2\sqrt{2c_2} \right), \quad (38)$$

$$\tau = Q_1 - \frac{1}{3}(P_1-1) - \frac{\sigma}{3}(P_1-1)^2. \quad (39)$$

If the universe expands without limit, $a \rightarrow \infty$, $t \rightarrow \infty$, then Eq. (36) is always satisfied, and P_1, Q_1 are the asymptotic values of P, Q . For $\mathcal{C} > 0$, the variable P plays the role of a time parameter, since $P(a)$ is monotonically decreasing, with $P > P_1$. Thus, in (P, Q) -space, the cosmological evolution is determined by the curve

$$Q(P) = Q_1 + \frac{P-P_1}{3} + \frac{\sigma}{3} \left[(P-1)^2 + (P_1-1)^2 \right]. \quad (40)$$

There is a well-defined cosmological evolution when this curve passes through the region defined by the inequalities (22) and (23), which in turn

depends on the values of the parameters P_1 , Q_1 and σ . A discussion analogous to the previous one is also valid for $\mathcal{C} = 0$.

One can verify that for $\mathcal{C} \geq 0$ there is a region of parameter space for which f is positive. The solutions with $0 < f_0 < \infty$ represent models that *avoid a cosmological singularity (in density and curvature), and undergo accelerated expansion from a_0* . Furthermore, for $\mathcal{C} > 0$, there are solutions which have infinite acceleration at a_0 .

The bulk black hole is crucial to the possibility of infinite acceleration. For $\mathcal{C} = 0$, one can show that f_0 cannot become $+\infty$ for a radiation era. We also note that all the above results hold independently of the spatial curvature k of the universe.

The accelerating expansion at and near a_0 , that is driven by geometric effects, serves as a “geometric” form of inflation, very different from conventional scalar field inflation. This could be interpreted as an alternative to inflaton scenarios, based on a quantum-gravity correction. However, there remain two crucial caveats.

(1) For $k \geq 0$, there is *no exit* from acceleration for the range of accelerating parameters σ , τ which give infinite f_0 , in the radiation era. This can be seen, using Eq. (24), from the fact that the sum of the first two terms in Eq. (34) is always positive, since $P^2 - 6Q$ is monotonically decreasing, and the last term in Eq. (34), proportional to $\dot{\Theta}$, is also positive. The range of σ, τ values gives only a sufficient condition for acceleration, and we have not been able to characterize the whole parametric space $(P_1, Q_1, \sigma, \beta)$. Therefore, it is still possible that some parameters exist that lead to an exit from inflation.

(2) Those solutions with $f_0 \rightarrow \infty$ have a *divergence of the Ricci scalar R on the brane, even though the density is finite*. This is impossible in general relativity or the Randall-Sundrum model, since in both cases $R = -T$, where T is the trace of the brane energy momentum tensor. This simple relation breaks down when there are curvature corrections, and the bulk curvature, interacting with the brane curvature and matter, plays a decisive role. Thus, the minimal epoch a_0 marks a curvature singularity, and the brane spacetime geometry breaks down there.

The acceleration-deceleration behaviour of the pure Gauss-Bonnet and the pure induced gravity models, is very different. For the Gauss-Bonnet case, we find from Eq. (10) that for $\mathcal{C} > 0$ it is

$$f = -\frac{1}{4\alpha} + \frac{1}{16\alpha} \left(1 - \frac{64I^2}{J^2}\right) \left(2J + \frac{J}{H}\right) + \frac{16\tilde{c}_2}{\alpha J}, \quad (41)$$

where

$$\begin{aligned} \sqrt{J} \frac{\dot{J}}{H} = & - \frac{2\tilde{\sigma}(1+w)(I^2 - \tilde{c}_2)^{3(1+w)/4}}{\sqrt{[\tilde{c}_1 + \tilde{\sigma}(I^2 - \tilde{c}_2)^{3(1+w)/4}]^2 + (8I)^3}} \times \\ & \times \left[J^{3/2} + \frac{512I}{\tilde{\sigma}(1+w)}(I^2 - \tilde{c}_2)^{(1-3w)/4} \right], \end{aligned} \quad (42)$$

with $\tilde{c}_1 = \sqrt{\alpha}\kappa_5^2\lambda/\sqrt{2}$, $\tilde{c}_2 = (3+4\alpha\Lambda_5)/192$, and $\tilde{\sigma} = (\sqrt{\alpha}\kappa_5^2\rho_0/\sqrt{2})(8/\alpha\mathcal{C})^{3(1+w)/4}$. In (I, J) -space, the curve defining the evolution of the Gauss-Bonnet universe is

$$\begin{aligned} J(I) = & \left\{ \tilde{c}_1 + \tilde{\sigma}(I^2 - \tilde{c}_2)^{3(1+w)/4} \right. \\ & \left. + \sqrt{[\tilde{c}_1 + \tilde{\sigma}(I^2 - \tilde{c}_2)^{3(1+w)/4}]^2 + (8I)^3} \right\}^{2/3}. \end{aligned} \quad (43)$$

Here, I can play the role of time parameter, with $I(a)$ monotonically decreasing for $\mathcal{C} > 0$. In the case $\mathcal{C} = 0$, similar expressions are valid. The only candidate quantity in Eq. (41) for producing divergence in f is the term $2J + \dot{J}/H$. By carefully examining the various situations, we obtain the result that in the radiation era of the Gauss-Bonnet universe there is no infinite acceleration. In the combined theory, because there is no infinite-density regime, the early universe behaviour cannot be obtained by expanding for large ρ . On the contrary, in the pure Gauss-Bonnet theory, equation (27) is the large ρ expansion, from which we can see furthermore that *there is no initial acceleration, and for $\rho \rightarrow \infty$, $R \rightarrow +\infty$.*

Finally, the induced gravity equation (14) gives in the radiation era:

$$\begin{aligned} f(\rho) = & \frac{\kappa_4^2}{3}(\lambda - \rho) + \frac{2}{r^2} \\ & \pm \frac{\sqrt{2}}{\sqrt{3}r} \left(2\kappa_4^2\lambda - \Lambda_5 + \frac{6}{r^2} \right) \times \\ & \times \left\{ 2\kappa_4^2 \left[\lambda + \left(1 - \frac{3\mathcal{C}}{\kappa_4^2\rho_0} \right) \rho \right] - \Lambda_5 + \frac{6}{r^2} \right\}^{-1/2}. \end{aligned} \quad (44)$$

We see from Eqs. (14) and (44) that among the solutions of the induced gravity model, there are some which start with initial singularity $a = 0$, as in the conventional model with $f = -\infty$. Moreover, there is at least one family of solutions for the branch with the $+$ sign, characterized by the conditions $2\kappa_4^2\lambda - \Lambda_5 + 6/r^2 > 0$, $\kappa_4^2\rho_0 < 3\mathcal{C}$ and $k \leq 0$, which start at a finite scale

factor with infinite acceleration, qualitatively similar to our model. Adopting the point of view that the characteristics of infinite-density avoidance and initial infinite acceleration are interesting cosmological features which are still present in the combined induced gravity plus Gauss-Bonnet model, we can say that the inclusion of the Gauss-Bonnet term has improved the situation by eliminating all the infinite-density solutions.

3.3 Late universe

For the parameters that allow $a \rightarrow \infty$, Eq. (24) is approximated as

$$H^2 + \frac{k}{a^2} \approx \frac{4 - 3\beta - \gamma}{12\beta\alpha} + \nu\kappa_4^2\rho, \quad (45)$$

neglecting terms $O(\rho^{4/3})$, where the dimensionless parameters γ and ν are

$$\gamma = 8\sqrt{P_1^2 - 6Q_1} \cos\left(\Theta_1 \pm \frac{\pi}{3}\right), \quad (46)$$

$$\begin{aligned} \nu = & \frac{2}{3\sqrt{P_1^2 - 6Q_1}} \left[\cos\left(\Theta_1 \pm \frac{\pi}{3}\right) + \sin\left(\Theta_1 \pm \frac{\pi}{3}\right) \times \right. \\ & \left. \times \frac{3Q_1 + 2P_1(1 - P_1)}{\sqrt{3}\sqrt{4Q_1(9P_1 - 8) + 4P_1^2(1 - P_1) - 27Q_1^2}} \right], \end{aligned} \quad (47)$$

with $\Theta_1 = \Theta(P_1, Q_1)$.

First, we observe that the bulk black hole mass \mathcal{C} does not appear, which means that even if it is non-zero, it decouples during the cosmological evolution and does not affect the late universe dynamics. The bulk is felt in the late universe only through its vacuum energy Λ_5 .

Second, for the branch with the $+$ sign in Eq. (24), because of the inequalities (22), (23), it follows that $\nu > 0$. Thus, although the last term in Eq. (24) is negative, in the late-time limit it produces both a negative cosmological constant, $-\gamma$, which contributes to the total cosmological constant, and a linear ρ term with positive Newton constant. For the branch with the $-$ sign, ν may be negative. Third, it is seen from the $a \rightarrow \infty$ limit of Eq. (24) that the quantity $4 - 3\beta - \gamma$ is always non-negative. Therefore, the conventional cosmology is recovered with positive effective gravitational and cosmological constants:

$$G_{\text{eff}} = 3\nu G_4, \quad \Lambda_{\text{eff}} = \frac{4 - 3\beta - \gamma}{4\beta\alpha}. \quad (48)$$

In the Gauss-Bonnet case, the late-universe limit of Eq. (10) is

$$H^2 + \frac{k}{a^2} \approx \frac{\Lambda_{\text{eff}}}{3} + \frac{8\pi G_{\text{eff}}}{3} \rho, \quad (49)$$

neglecting terms $O(\rho^{4/3})$, where the effective constants are

$$\Lambda_{\text{eff}} = \frac{3}{8\alpha} \left(-2 + \frac{64I_1^2}{J_1} + J_1 \right), \quad (50)$$

$$G_{\text{eff}} = \frac{G_5}{2\sqrt{2\alpha}} \sqrt{J_1} \left[\frac{J_1^2 - (8I_1)^2}{J_1^3 + (8I_1)^3} \right]. \quad (51)$$

Here I_1, J_1 are the asymptotic values for $a \rightarrow \infty$ of the variables I, J , defined in terms of the parameters \tilde{c}_1, \tilde{c}_2 of the Gauss-Bonnet model by the relations $I_1 = \sqrt{\tilde{c}_2}$, $J_1 = [\tilde{c}_1 + \sqrt{\tilde{c}_1^2 + (64\tilde{c}_2)^{3/2}}]^{2/3}$. The previous remarks concerning the non-appearance of \mathcal{C} in the above equations, as well as the positivity of the effective Newton and cosmological constants, are still valid.

When the brane tension is zero, the Friedmann equation (45) recovers the standard general relativity behaviour, since the coefficient ν in Newton's constant remains positive and nonzero if we set $\lambda = 0$. Therefore, if both curvature corrections are combined, the conventional cosmology is recovered, even for a tensionless brane. On the contrary, in the pure Gauss-Bonnet equation (49), the brane tension is essential, since $\lambda = 0$ implies $G_{\text{eff}} = 0$. This is like the pure Randall-Sundrum case, where positive brane tension is necessary in order to recover the standard Friedmann equation [2, ?].

The late-time limit of the pure induced gravity Friedmann equation (14) gives the positive constants

$$\Lambda_{\text{eff}} = \kappa_4^2 \lambda + \frac{6}{r^2} \pm \frac{\sqrt{6}}{r^2} \sqrt{(2\kappa_4^2 \lambda - \Lambda_5) r^2 + 6}, \quad (52)$$

$$G_{\text{eff}} = G_4 \left[1 \pm \left\{ \frac{r^2}{6} (2\kappa_4^2 \lambda - \Lambda_5) + 1 \right\}^{-1/2} \right]. \quad (53)$$

When there is no brane tension, and even no bulk cosmological constant, general relativity is still recovered [5].

4 Conclusions

We studied the cosmology of a brane-world with curvature corrections to the Randall-Sundrum gravitational action, i.e. a four-dimensional curvature

term of induced gravity and a five-dimensional Gauss-Bonnet term. The fundamental parameters appearing in the model are: three energy scales, i.e. the fundamental Planck mass M_5 , the induced-gravity crossover energy scale r^{-1} , and the Gauss-Bonnet coupling energy scale $\alpha^{-1/2}$, and two vacuum energies, i.e. the bulk cosmological constant Λ_5 and the brane tension λ . These parameters determine the cosmological evolution of the brane universe.

We derived the Friedmann equation of the combined curvature effects, Eq. (24), which smoothly matches to the induced gravity equation when the Gauss-Bonnet term vanishes. This equation has a structure which is quite different from its two limiting forms. All the solutions of the cosmological model are of finite density, independently of the spatial curvature of the universe and the equation of state. This is remarkable, since the Gauss-Bonnet correction on its own dominates at early times and does not remove the infinite-density singularity, while the induced gravity correction on its own mostly affects the late-time evolution. However, the combination of these curvature corrections produces an “interaction” that is not obviously the superposition of their separate effects. In general terms, the early-universe behaviour is strongly modified by the effective coupling of the 5D curvature to the matter.

The late cosmological evolution of our model follows the standard cosmology, even for zero brane tension, with a positive Newton constant for one of the two branches of the solutions and positive cosmological constant.

Acknowledgments

This work was done in collaboration with G. Kofinas and R. Maartens and it is partially supported by NTUA research program “Thalis”.

References

- [1] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 4690 (1999) [hep-th/9906064].
- [2] P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, *Phys. Lett.* **477B**, 285 (2000) [hep-th/9910219].
- [3] C. Csaki, M. Graesser, C. Kolda and J. Terning, *Phys. Lett.* **462B**, 34 (1999) [hep-ph/9906513]; J. Cline, C. Grojean and G. Servant, *Phys. Rev. Lett.* **83**, 4245 (1999) [hep-ph/9906523].

- [4] H. Collins and B. Holdom, *Phys. Rev.* **D62**, 105009 (2000) [hep-ph/0003173].
- [5] G. Dvali, G. Gabadadze and M. Porati, *Phys. Lett.* **485B**, 208 (2000) [hep-th/0005016]; G. Dvali and G. Gabadadze, *Phys. Rev.* **D63**, 065007 (2001) [hep-th/0008054].
- [6] D. Lovelock, *J. Math. Phys.* **12**, 498 (1971).
- [7] B. Zwiebach, *Phys. Lett.* **156B**, 315 (1985); B. Zumino, *Phys. Rept.* **137**, 109 (1986); D.J. Gross and J.H. Sloan, *Nucl. Phys.* **B291**, 41 (1987).
- [8] Y. Shtanov, hep-th/0005193; S. Nojiri and S.D. Odintsov, *JHEP* **07**, 049 (2000) [hep-th/0006232]; C. Deffayet, *Phys. Lett.* **502B**, 199 (2001) [hep-th/0010186]; G. Kofinas, *JHEP* **08**, 034 (2001) [hep-th/0108013]; N.J. Kim, H.W. Lee and Y.S. Myung, *Phys. Lett.* **504B**, 323 (2001) [hep-th/0101091]; C. Deffayet, G. Dvali and G. Gabadadze, *Phys. Rev.* **D65**, 044023 (2002) [astro-ph/0105068]; C. Deffayet, S.J. Landau, J. Raux, M. Zaldarriaga and P. Astier, *Phys. Rev.* **D66**, 024019 (2002) [astro-ph/0201164].
- [9] K. Maeda, S. Mizuno and T. Torii, *Phys. Rev.* **D68**, 024033 (2003) [gr-qc/0303039].
- [10] C. Charmousis and J. Dufaux, *Class. Quantum Grav.* **19**, 4671 (2002) [hep-th/0202107].
- [11] E. Kiritsis, N. Tetradis and T.N. Tomaras, *JHEP* **03**, 019 (2002) [hep-th/0202037].
- [12] V. Sahni and Y. Shtanov, *Int. J. Mod. Phys.* **11**, 1 (2002) [gr-qc/0205111]; U. Alam and V. Sahni, astro-ph/0209443.
- [13] G. Kofinas, E. Papantonopoulos and I. Pappa, *Phys. Rev.* **D66**, 104014 (2002) [hep-th/0112019]; G. Kofinas, E. Papantonopoulos and V. Zamarias, *Phys. Rev.* **D66**, 104028 (2002) [hep-th/0208207]; A. Lue and G. Starkman, *Phys. Rev.* **D67**, 064002 (2003) [astro-ph/0212083]; G. Dvali, A. Gruzinov and M. Zaldarriaga, *Phys. Rev.* **D68**, 024012 (2003) [hep-ph/0212069].
- [14] G. Kofinas, R. Maartens and E. Papantonopoulos, *JHEP* **0310**, 066 (2003) [hep-th/0307138].

- [15] C. Germani and C. Sopuerta, Phys. Rev. Lett. **88**, 231101 (2002) [hep-th/0202060].
- [16] J.E. Lidsey and N.J. Nunes, Phys. Rev. **D67**, 103510 (2003) [astro-ph/0303168].
- [17] R. Maartens, D. Wands, B.A. Bassett and I.P.C. Heard, Phys. Rev. **D62**, 041301 (2000) [hep-ph/9912464].